Irrational Square Roots:

$\sqrt{2}$ and $\sqrt{n}$ when n is a Positive Integer and Not a Perfect Square

[It is recommended that you review Theorem (NIB) 3 in the handout "Theorems (NIB) 1,2, and 3."]

Theorem 4.6.1: $\sqrt{2}$ is irrational.

Proof: [Proof by Contradiction]

Suppose, by way of contradiction, that $\sqrt{2}$ is rational.

Since $\sqrt{2}$ is rational and positive, there exist positive integers m and n, with $n \neq 0$, such that $\sqrt{2} = \frac{m}{n}$, and we can assume that $\frac{m}{n}$ is written in lowest terms, so that m and n have no common prime factor.

[The author mistakenly says that m and n "have no common factor", but 1 is always a common factor.]

Since $\sqrt{2} = \frac{m}{n}$, $2 = \left(\sqrt{2}\right)^2 = \left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2}$ by substitution .

Since $2 = \frac{m^2}{n^2}$, $2n^2 = m^2$.

[The contradiction that we will establish is that $2 \mid m$ and $2 \mid n$,

which contradicts the fact that m and n have no common prime factor.]

Since $m^2 = 2n^2$ and n^2 is an integer, $2 \mid m^2$, by definition of "divides".

$\therefore$ Since $2 \mid m^2$ and 2 is prime, $2 \mid m$, by Theorem (NIB) 3 .

$\therefore$ There exists an integer k such that $m = 2k$, by definition of "divides". Recall that $2n^2 = m^2$.

$\therefore$ $2n^2 = (2k)^2 = 2(2k^2)$, by substitution and the rules of algebra.

Dividing by 2, we conclude that $n^2 = 2k^2$, and k^2 is an integer .

$\therefore$ $2 \mid n^2$, by definition of "divides".

$\therefore$ Since $2 \mid n^2$ and 2 is prime, $2 \mid n$, by Theorem (NIB) 3 .

$\therefore$ $2 \mid m$ and $2 \mid n$, which contradicts the fact that m and n have no common prime factors.

Therefore, $\sqrt{2}$ is irrational , by proof-by-contradiction

QED

[You might consider how this proof can be adapted to prove that $\sqrt{5}$ and $\sqrt{7}$ are irrational.]

To Prove: For all positive integers n , if n is not a perfect square, then $\sqrt{n}$ is irrational.

[This is the statement to be proved in Problem #22 of Section 4.6,]

Proof: [by Contraposition]

Let n be any positive integer.

Suppose that $\sqrt{n}$ is rational. [We need to show that n is a perfect square.]

Since $\sqrt{n}$ is rational and positive, there exist positive integers a and b with $b \neq 0$ such that $\sqrt{n} = \frac{a}{b}$,

and we can assume that $\frac{a}{b}$ is written in lowest terms, so that a and b have no common prime factor.

Since $\sqrt{n} = \frac{a}{b}$, $n = \left(\sqrt{n}\right)^2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$. Since $n = \frac{a^2}{b^2}$, $b^2 n = a^2$.

[We next prove that b = 1 using a proof-by-contradiction .]

Suppose, by way of contradiction, that $b \neq 1$. (***)

$\therefore$ Since $b > 0$ and $b \neq 1$, $b > 1$.

$\therefore$ by Theorem 4.3.4 , there exists some prime number p such that $p \mid b$.

Since $b^2 n = b(bn)$, $b \mid b^2 n$ by definition of "divides" .

$\therefore$ $p \mid b^2 n$, by transitivity of divisibility. Recall that $b^2 n = a^2$.

$\therefore$ $p \mid a^2$, by substitution.

$\therefore$ Since p is prime and $p \mid a^2$, $p \mid a$, by Theorem (NIB) 3 .

$\therefore$ $p \mid a$ and $p \mid b$, which contradicts the fact that a and b have no common prime factor.

$\therefore$ $b = 1$ by proof-by-contradiction. [Considering the initial supposition (***) above]

$\therefore$ $n = \frac{a^2}{b^2} = \frac{a^2}{1} = a^2$, and, therefore, n is a perfect square.

$\therefore$ If n is not a perfect square, then $\sqrt{n}$ is irrational, by contraposition.

$\therefore$ For all positive integers n, if n is not a perfect square, then $\sqrt{n}$ is irrational, by Direct Proof .

QED

[When applying this result, use the justification , "by Problem #22 of Section 4.6."]